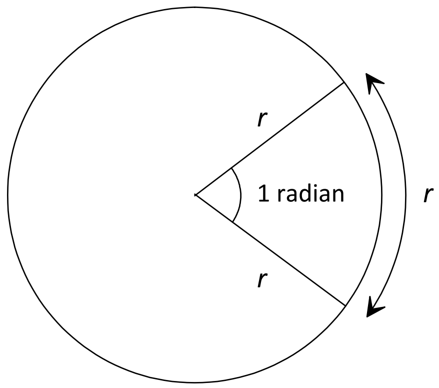
Radians

Radians are a way of measuring angles. Suppose a circle has a radius of r, then one radian is the angle at the centre of a circle opposite the arc of length r and between two radii. More formally, one radian is defined as 'the angle subtended at the centre of an arc of length r'. On a diagram this is:



Since the circumference of a circle of radius r is equal to $2\pi r$, it means that there will be 2π radians in a full circle:

$$2\pi \text{ radians} = 360^{\circ}$$

$$\pi \text{ radians} = 180^{\circ}$$

Trigonometry

The standard trigonometric functions of sine, cosine and tangent are defined as the following ratios:

$$\sin x = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos x = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan x = \frac{\text{opposite}}{\text{adjacent}}$$

Their reciprocals give the cosecant, secant and cotangent:

$$\operatorname{cosec} x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

Example

Calculate:

(i) $\cot\frac{\pi}{6}$ (ii) the value of θ where $\operatorname{cosec}\theta = \frac{4}{3}$

Solution

(i) We have:

$$\cot\frac{\pi}{6} = \frac{1}{\tan\frac{\pi}{6}} = \sqrt{3} = 1.73205$$

(ii) We have:

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{4}{3} \;\Rightarrow\; \sin\theta = \frac{3}{4} \;\Rightarrow\; \theta = 0.84806 \text{ radians}$$

Differentiation of trig functions

The standard results are shown in the table below:

Function	***Derivative***
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$

The proof of the third result can be obtained using the first two results and the quotient rule. We can then use these results in the chain rule as the next example shows:

Example

Find the derivative of:

(i) $y = \cos 3x$ (ii) $y = \sqrt{\tan x}$

Solution

(i) Setting $u = 3x$ we have $y = \cos u$. Hence:

$$\frac{dy}{du} = -\sin u \qquad \frac{du}{dx} = 3$$

This gives:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\sin u \times 3$$

Replacing u gives:

$$\frac{dy}{dx} = -3\sin 3x$$

(ii) Rewriting this as $y = (\tan x)^{½}$ and setting $u = \tan x$ we have $y = u^{½}$. Hence:

$$\frac{dy}{du} = ½u^{-½} \qquad \frac{du}{dx} = \sec^2 x$$

This gives:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = ½u^{-½} \times \sec^2 x$$

Replacing u gives:

$$\frac{dy}{dx} = ½\sec^2 x(\tan x)^{-½} = \frac{\sec^2 x}{2\sqrt{\tan x}}$$

Question 1.1

Differentiate the following:

(i) $y = \sin^2 x$ (ii) $y = \tan(5x+1)$ (iii) $y = \sec x$

Trigonometric functions could also form part of a product or quotient so we will need to be able to apply the product and quotient rules.

Example

Find the derivative of:

(i) $y = x\tan(\frac{1}{2}x)$ (ii) $y = \dfrac{\sin x}{\cos x}$

Solution

(i) Setting $u = x$ and $v = \tan(\frac{1}{2}x)$ we have:

$$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = \frac{1}{2}\sec^2(\tfrac{1}{2}x)$$

This gives:

$$\frac{dy}{dx}=\frac{du}{dx}v+u\frac{dv}{dx}=\tan(\tfrac{1}{2}x)+\tfrac{1}{2}x\sec^2(\tfrac{1}{2}x)$$

(ii) Setting $u=\sin x$ and $v=\cos x$ we have:

$$\frac{du}{dx}=\cos x \qquad \frac{dv}{dx}=-\sin x$$

This gives:

$$\frac{dy}{dx}=\frac{\frac{du}{dx}v-u\frac{dv}{dx}}{v^2}=\frac{\cos x\cos x-\sin x(-\sin x)}{(\cos x)^2}=\frac{\sin^2 x+\cos^2 x}{\cos^2 x}$$

Whilst this is the correct answer we can simplify this using the result $\sin^2 x+\cos^2 x=1$ which gives:

$$\frac{dy}{dx}=\frac{1}{\cos^2 x}=\sec^2 x$$

Hence we have proved the result $\frac{d}{dx}\tan x=\sec^2 x$.

Question 1.2

Differentiate the following:

(i) $y=x^2\sin 4x$ (ii) $y=\dfrac{\cos(1-2x)}{x}$

A useful application of trigonometric functions is in parametric equations where the x and the y are both defined in terms of another variable. This allows us to draw some pictures that would not be possible otherwise.

Example

A unit circle can be drawn by the parametric equations:

$$y = \sin\theta \qquad x = \cos\theta$$

Obtain the derivative dy/dx.

Solution

We have:

$$\frac{dy}{d\theta} = \cos\theta \qquad \frac{dx}{d\theta} = -\sin\theta$$

Hence:

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} = \frac{\cos\theta}{-\sin\theta} = -\frac{1}{\tan\theta} = -\cot\theta$$

Limits

One of the questions on the sample ACET paper requires the use of L'Hôpital's rule to find the limit of a function. This is given as follows:

Let $f(x)$ and $g(x)$ be functions such that $f(x_0) = g(x_0) = 0$, but $f'(x_0)$ and $g'(x_0)$ are not both zero. Then:

$$\lim_{x \to x_0} \left(\frac{f(x_0)}{g(x_0)} \right) = \frac{f'(x_0)}{g'(x_0)}$$

Should both of the derivatives be zero at x_0 then this rule can be expanded to higher derivatives.

Essentially this rule is saying that if we want to find the limit of a function which is going to give us $\frac{0}{0}$ then we differentiate the numerator and denominator until we get something where both are not zero. That will give us our limit.

Here's an example:

Example

Find $\lim_{x \to 0}\left(\frac{\sin x}{x}\right)$.

Solution

As both $\sin x \to 0$ and $x \to 0$ we have the problem of determining the limit of $\frac{0}{0}$. So we need to make use of L'Hôpital's rule. Differentiating the numerator $f(x)=\sin x$ and the denominator $g(x)=x$ we get:

$$f'(x)=\cos x \qquad g'(x)=1$$

Now these are not both zero when $x \to 0$ (in fact neither of them are), so we can say that the limit will be equal to:

$$\lim_{x \to 0}\left(\frac{\cos x}{1}\right)=\frac{1}{1}=1$$

Question 1.3

Find $\lim_{x \to 0}\left(\frac{x^2}{\tan x}\right)$.

Solutions

Solution 1.1

(i) Recall that $\sin^2 x$ is shorthand notation for $(\sin x)^2$. Setting $u = \sin x$ gives $y = u^2$. Hence:

$$\frac{dy}{du} = 2u \qquad \frac{du}{dx} = \cos x$$

This gives:

$$\frac{dy}{dx} = 2u \times \cos x = 2\sin x \cos x$$

(ii) Setting $u = 5x + 1$ gives $y = \tan u$. Hence:

$$\frac{dy}{du} = \sec^2 u \qquad \frac{du}{dx} = 5$$

This gives:

$$\frac{dy}{dx} = \sec^2 u \times 5 = 5\sec^2(5x + 1)$$

(iii) Recall that $\sec x = \frac{1}{\cos x} = (\cos x)^{-1}$. Setting $u = \cos x$ gives $y = u^{-1}$. Hence:

$$\frac{dy}{du} = -u^{-2} \qquad \frac{du}{dx} = -\sin x$$

This gives:

$$\frac{dy}{dx} = -u^{-2} \times -\sin x = \frac{\sin x}{\cos^2 x}$$

Solution 1.2

(i) Setting $u=x^2$ and $v=\sin(4x)$ we have:

$$\frac{du}{dx}=2x \qquad \frac{dv}{dx}=4\cos(4x)$$

This gives:

$$\frac{dy}{dx}=\frac{du}{dx}v+u\frac{dv}{dx}=2x\sin 4x+4x^2\cos 4x$$

(ii) Setting $u=\cos(1-2x)$ and $v=x$ we have:

$$\frac{du}{dx}=+2\sin(1-2x) \qquad \frac{dv}{dx}=1$$

This gives:

$$\frac{dy}{dx}=\frac{\frac{du}{dx}v-u\frac{dv}{dx}}{v^2}=\frac{2x\sin(1-2x)-\cos(1-2x)}{x^2}$$

Solution 1.3

As both $x^2\to 0$ and $\tan x\to 0$ we have the problem of determining the limit of $\frac{0}{0}$. So we need to make use of L'Hôpital's rule. Differentiating the numerator $f(x)=x^2$ and denominator $g(x)=\tan x$ we get:

$$f'(x)=2x \qquad g'(x)=\sec^2 x$$

Now these are not both zero when $x\to 0$ (only the first one is), so we can say that the limit will be equal to:

$$\lim_{x\to 0}\left(\frac{2x}{\sec^2 x}\right)=\frac{0}{1}=0$$

Integration of trig functions

Integration of trigonometric functions is just the inverse of differentiating. So we have:

Function	**Integral**
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\sec^2 x$	$\tan x$

Example

Calculate $\int_0^{\pi/4} \sin(3x+\frac{\pi}{2})\,dx$

Solution

Setting $u=3x+\frac{\pi}{2}$ we have $\frac{du}{dx}=3$. The limits from $x=0$ to $x=\frac{\pi}{4}$ become $u=\frac{\pi}{2}$ to $u=\frac{5\pi}{4}$. We now have:

$$\int_{\pi/2}^{5\pi/4} \frac{1}{3}\sin u\,du = \left[-\frac{1}{3}\cos u\right]_{\pi/2}^{5\pi/4} = -\frac{1}{3}\left(\cos\frac{5\pi}{4}-\cos\frac{\pi}{2}\right)=\frac{\sqrt{2}}{6}=0.23570$$

Trig functions can also be used in integration by parts:

Example

Calculate $\int_{-\pi/4}^{\pi/4} x\cos x\,dx$.

Solution

Let $u = x$ and $\frac{dv}{dx} = \cos x$. This gives $\frac{du}{dx} = 1$ and $v = \sin x$. Using the integration by parts formula we have:

$$\int_{-\pi/4}^{\pi/4} x\cos x\,dx = \left[x\sin x\right]_{-\pi/4}^{\pi/4} - \int_{-\pi/4}^{\pi/4} \sin x\,dx$$

$$= \left[x\sin x\right]_{-\pi/4}^{\pi/4} - \left[-\cos x\right]_{-\pi/4}^{\pi/4}$$

$$= \frac{\pi}{4}\sin\frac{\pi}{4} + \frac{\pi}{4}\sin\left(-\frac{\pi}{4}\right) + \cos\frac{\pi}{4} - \cos\left(-\frac{\pi}{4}\right)$$

$$= 0$$

Question 1.1

Determine:

(i) $\int \sec^2(2x)\,dx$

(ii) $\int_0^{\pi} x\sin(\tfrac{1}{2}x)\,dx$

(iii) $\int \tan x\,dx$ (Hint: use the substitution $u = \cos x$)

Solutions

Solution 1.1

(i) Setting $u=2x$ we have $\dfrac{du}{dx}=2$. We now have:

$$\int \tfrac{1}{2}\sec^2 u\,du = \tfrac{1}{2}\tan u = \tfrac{1}{2}\tan(2x)+c$$

(ii) Differentiating x we get 1 which is simpler than differentiating the sine function. So let $u=x$ and $\dfrac{dv}{dx}=\sin(\tfrac{1}{2}x)$. This gives $\dfrac{du}{dx}=1$ and using integration by substitution $v=-2\cos(\tfrac{1}{2}x)$. Using the integration by parts formula we have:

$$\begin{aligned}\int_0^{\pi} x\sin(\tfrac{1}{2}x)\,dx &= \left[-2x\cos(\tfrac{1}{2}x)\right]_0^{\pi} + \int_0^{\pi} 2\cos(\tfrac{1}{2}x)\,dx \\ &= \left[-2x\cos(\tfrac{1}{2}x)\right]_0^{\pi} + \left[4\sin(\tfrac{1}{2}x)\right]_0^{\pi} \\ &= -2\pi\cos\frac{\pi}{2} + 0 + 4\sin\frac{\pi}{2} - 4\sin 0 \\ &= 4\end{aligned}$$

(iii) Rewriting, we have:

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

Setting $u=\cos x$ we have $\dfrac{du}{dx}=-\sin x$. We now have:

$$-\int \frac{1}{u}\,du = -\ln u + c = -\ln(\cos x)+c$$

